# Attractors and Nonlinear Dynamical Systems

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The harmony of the world is made manifest
in Form and Number, and the heart and soul
and all the poetry of Natural Philosophy
are embodied in the concept of mathematical beauty.

*-D'Arcy Wentworth Thompson, On Growth and Form*

The mathematical object known as an *attractor* is central to the field of mathematics known as *nonlinear dynamical systems theory* (NDS), one of the indispensible conceptual underpinnings of complexity science. To appreciate what an attractor is accordingly demands some familiarity with other NDS notions, namely, *phase* or *state space*, *phase portraits*, *basins of attraction*, *initial conditions*, *transients*, *bifurcations*, *chaos*, and *strange attractors*. These terms may seem imposingly technical at first but in effect they turn out to be relatively simple in their proven capacity for taming some of the unruliness of complex systems. If all we had available in understanding complex systems were theoretical constructs of the same or even greater complexity, we would be unable to make much headway in either comprehending the dynamics of complexity or in utilizing what we know.

Most of us have by now at least some inkling of what "nonlinear" means. Mathematically, it refers to *dis*proportional relationships among variables in equations and, accordingly, the systems represented by those equations and variables, for example, what occurs whenever there is some kind of mutual interaction or feedback going on among the variables. Perhaps, the most well-known and vivid example of this kind of nonlinear *dis*proportionality is the "butterfly effect" of a technically *chaotic* system that is so nonlinear it has prompted the use of the image of tiny air currents produced by a butterfly flapping its wings in Brazil, which are then amplified to the extent they may influence the building-up of a thunderhead in Kansas. To be sure, no one has actually claimed there is such a linkage between Brazilian lepidopterological dynamics and climatology in the Midwest of the USA but it does serve to vividly portray nonlinearity *in extremis*.

> Perhaps, the most well-known and vivid example of nonlinear disproportionality is the "butterfly effect."

A *nonlinear dynamical system* is one that unfolds in a law-like manner, that is, one customarily distinguished from a totally random system (although randomization can play a significant role in certain dynamical systems) but in which outcomes are nevertheless unpredictable in important respects because of both the nonlinearity and the capacity of such systems in passing through different regimes of stability and instability. These different regimes of a dynamical system are understood as different *phases* "governed" by a different attractor(s). "Governed" is used here in a loose sense, the idea being that the dynamics of each phase of a dynamical system are constrained within the circumscribed range allowable by that phase's attractor(s). It is not untypical to hear the term "attractor" being used in the sense of possessing some sort of causal efficacy (a topic we'll get to later). In actuality, an attractor is an abstract mathematical representation and not a cause *per se* "existing" in an abstract mathematical space termed "phase" or "state space", composed of the values of the variables of interests plotted against each other and which may exhibit particular "phase portraits" that reveal important aspects of the dynamics of such systems.

## Time Series

To better grasp the idea of phase space, let's contrast it with a different way of representing the change of a system over time: a *time series* chart that plots the changing values of the variable(s) on the y-(or z- or...) axis and time on the x-axis. Amidst the financial crisis that came to a head at the end of 2008, we unfortunately became all too familiar with the downward zigzagging of capital market time series charts. For example, here is a dismal example of General Motors' stock price on the y-axis over the period on the x-axis, ending in November, 2008:

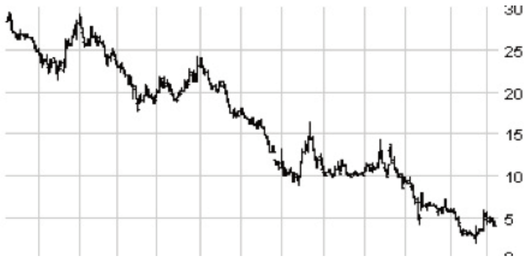


*Figure 1: General Motors Stock Prices for 12 months to Nov. 2008 (GM, No date).*

From looking at this time series chart alone, we can appreciate why GM took so much heat, particularly in the context of exorbitant bonuses and extravagant perks given over this same time period. If this were the times series chart for, say, revenue of a small business, it would presage bankruptcy, not extra income. However, if we compare this time series for the year up to November 2008 with the recent IPO of GM, we can recognize one of the limitations of using time series charts in reaching dogmatic conclusions.

Here is another example, this one of gold prices (considered a good investment by those who tend not to trust anything unless it glitters in their hands. But alas, so does fool's gold!):


*Figure 2: Gold Prices for a Year (Gold Prices, 2008).*

To be sure, this time series chart does indicate that gold was a better investment for the same year than GM stock for the same time period. However, juxtaposing the two times series charts of Figures 1 and 2 points to another of the limitations of relying on time series alone. Consider the following facts. In 1990 gold sold for $380 and Figure 2 shows that gold was selling on Dec. 5th for about $760. This means that investors who bought gold in 1990 just about doubled their investment in 18 years, a seemingly not too bad 200% increase. However, it is also true that the currently much bemoaned S&P index was about $350 in 1990 and even after its devaluation due to the meltdown (no pun intended) of 2008, was close to $900. Thus, an S&P index fund would show an increase of $350 to $900 which translates into a 257% increase against a mere 200% increase in gold. Besides the obvious conclusion that mixing political leanings with investing is a fool's errand, the comparison of Figures 1 and 2 reveals that times series are always of a very limited time horizon and whatever trends are seen in some segment of that time horizon do not necessarily translate to other segments of time or to longer series.

This limitation of time series is also demonstrated in Figure 3, a times series chart for retail gasoline prices over the past eight years ending on Dec. 1, 2008. We see a zigzagging but overall strong increase for most of this chart, a literal "going through the roof," until suddenly, during the Fall of 2008, the "bottom dropped out" of the oil price boom and accordingly, gasoline prices as well. But, this sudden and precipitous drop was nowhere evident in the time series chart for the preceding eight years:

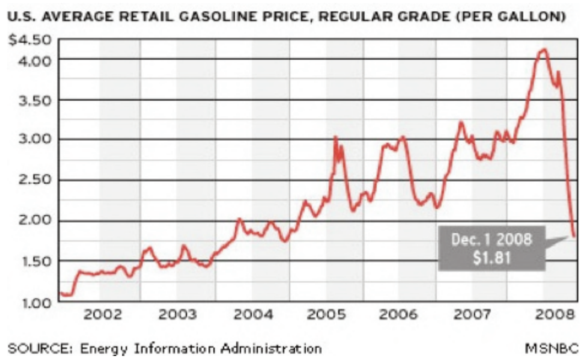


*Figure 3: Price of Retail Gasoline per gallon since 2001 (MSNBC, 2008).*

Despite such serious disadvantages, financial time series charts are carefully inspected everyday by "technical analysis" in the attempt to predict the direction of their zigzagging lines. These "chartists" have identified such supposedly repeating and therefore stable patterns as "double-bottoms," "hanging man lines," "the morning star," "head and shoulders," and so on (Sornette, 2003). Although research has shown that "head and shoulders" does, to some extent, surprisingly correlate with predictions of prices, this correlation is hypothesized to be due to self-fulfilling prophecies on the part of investors committed to investing when they see this pattern (see Osler & Chang, No date). A blind belief, however, in such purportedly repeating patterns in time series market charts is pretty much the same as believing that the stars and galaxies that make celestial constellations have a special relationship with each because they are part of these constellations.

One of the most egregious examples, in my opinion, that demonstrates just how far time series "chartists" can go in detecting patters is  the "Elliott Wave." Proponents claim it is present in most if not all capital market charts.

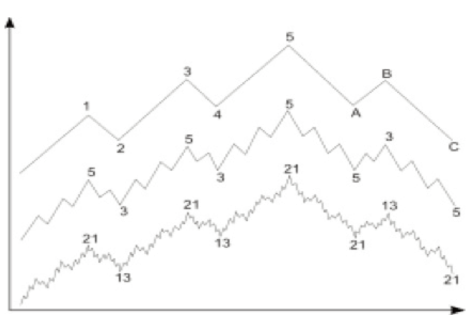


*Figure 4: Elliott Waves in Stock Price Time Series Charts (Elliott Waves, No date).*

At early conferences on chaos theory and NDS, Elliott Wave aficionados would show up with their claims of finding "Elliott Waves" in scientific data from many fields. However, in the face of doubters who didn't see the Elliott Waves where these acolytes pointed, the latter would respond by saying that these obdurate doubters were not looking at the time series charts in the right way. No doubt they were correct in that assessment, because to see Elliott Waves everywhere one first needs to be a "true believer."

Times series charts do play an important part in complexity science. One salient example is the very sophisticated "discrete scale invariance" found in time series charts devised by the complexity-based geophysicist Didier Sornette (2003). Sornette propounds a mathematical approach to chart analysis that relies on elements of fractal geometry, the theory of self-organized criticality, power law statistics and other complexity constructs. Although Sornette was able to "predict," by hindsight, certain market crashes in Asia, as well as by retrodiction, the crashes in the US in 1929 and 1987, his predictions for a major capital market meltdown for 2003 or 2004 didn't materialize until four years later. One wonders if there isn't a bit of the "a broken clock tells the correct time two times per day."

## Phase Space, Phase Portraits, and Attractors

Instead of relying only on times series charts, dynamical systems researchers appeal to a different mathematical representation of data points, the *phase portrait* displayed in *phase* or *state space* (see Fogelberg, 1992) Rather than plotting changes in the values of variables on the **y**-axis (or **z**-axis), and time on the ***x***-axis as in a time series chart, a phase space diagram plots the variables against each other and leaves time as an implicit dimension not explicitly graphed.

The origins of the idea of phase space are a bit hazy, but it was a very important tool used to simplify dynamics used in the mid-nineteenth century by the very influential but not well-known American physicist J. Willard Gibbs. When the data is plotted in phase space with points in phase space representing the value of each of the variables at each moment of time, as the system changes over time, the data points make up a trajectory that is called a *phase portrait*. Certain phase portraits then display attractor(s) as the long-term stable sets of points of the dynamical system, that is, the locations in the phase portrait towards which the system's dynamics are attracted after transient phenomena have died down. Since phase space and attractors are abstract mathematical objects, some concrete examples can help in understanding what's going on.

Imagine a child on a swing (a type of pendulum) and a parent pulling the swing back and giving it a good push but then backing away. Also, assume the child is not moving forward or backward on the swing to influence its momentum (which would be going against the main fun for being on a swing). What eventually happens to the "unpushed" swing? It will come to rest. Here is a time series chart showing that eventuality.

The times series in Figure 1(b) shows an oscillation of the speed of the swing, which slows down and eventually stops, that is, it "flat lines." But we can also plot the same movement of the "unpushed" swing in phase space to generate a phase portrait. In phase space, the swing's speed from the central resting spot is plotted, not against time, but against the distance of the swing from the central resting place:

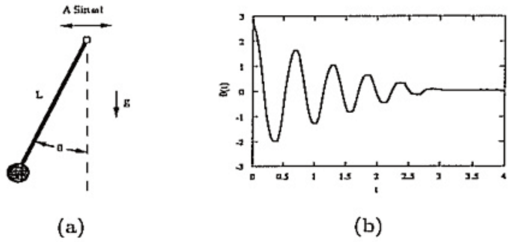


*Figure 5: (a) Drawing of an "Unpushed" Swing; (b) Time Series of the "Unpushed" Swing (y-axis is the velocity of the swing, positive value in one direction, negative in the other; the x-axis is time) (adapted from Bayly & Virgin, 1992).*

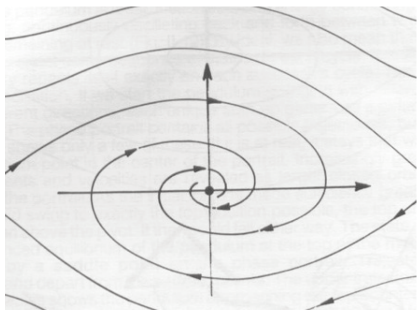

*Figure 6. Phase Portrait and Fixed Point Attractor of an "Unpushed" Swing (adapted from Abraham, 1982).*

The phase portrait in Figure 6 shows curved lines with arrows spiraling in toward a central point called a fixed point attractor since, it is attracting the system's dynamics in the long run. The fixed point attractor in the center of Figure 6 is equivalent to the flat line in Figure 5 (b).

The point in the center of the phase space diagram is an attractor, more specifically, a fixed point attractor, and an *attractor* since it represents the long term, stable pattern of the dynamical system of the "unpushed" swing over time after the transient and diminishing back and forth motion has died down. As an *attractor* of the unpushed swing, it displays what the dynamics of system are attracted to over the long run, as long as it doesn't receive any further pushes. The fixed point attractor is another way of seeing and saying that an "unpushed" swing will come to a state of rest in the long term. Now by itself, this may not seem particularly revealing. We need to wait for more complex attractors for that.

The curved lines with arrows spiraling down to the center point in the phase space diagram of Figure 6 display what is called the *basin of attraction* for the "unpushed" swing. These basins of attraction represent various *initial conditions* for the "unpushed" swing, that is, the starting heights and initial velocities. The lines spiral in since the heights of the swing and its speed slow down to zero when the dynamics reach the fixed point attractor. Basins of attraction can be likened to an actual basin or bowl of a sink with a drain at the bottom. The drain at the bottom of the sink is analogous to the fixed point attractor in the center of the phase space diagram. Wherever water is poured into the bowl of a sink, high up in the sink, half-way down, or even lower still,  and whatever the initial condition of where the water is when it starts its downward spiral, the water  will be eventually be drawn or "attracted" to go down the drain.

Now let's consider another type of a similar dynamical system, this time a "pushed" swing in which the parent keeps pushing the swing each time it comes back to where the parent is standing. A times series chart of the "pushed" swing, with time, as in the previous time series charts, represented on the ***x***-axis and the distance of the swing from a state of rest plotted as a value on the ***y***-axis, is shown in Figure 7 as a continuing oscillation around the ***x***-axis. This oscillation is around a zero value for ***y*** (the low point of the swing) and is positive when the swing is going in one direction and negative when the swing is going in the other direction:

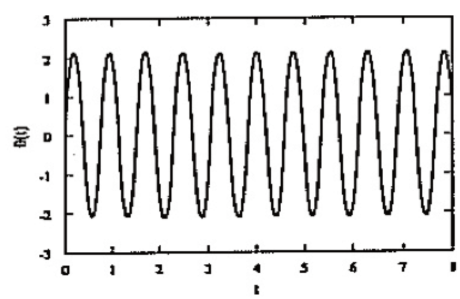


*Figure 7: Time Series Chart of the "Pushed" Swing (a continuing oscillation around the x-axis — adapted from Bayly & Virgin, 1992).*

Now let's convert this same scenario of the "pushed" swing to a phase diagram in phase space as we did in the case of the "unpushed" swing. As a phase space diagram, we plot the variables against each other, that is  the speed of the swing and the distance from the central resting point (again, time is an *implicit* dimension not directly plotted in phase space).

*Figure 8: Phase Space Diagram and Limit Cycle Attractor of a "Pushed" Pendulum (adapted from Abraham, 1982).*

The unbroken oval to which both the inner and outer lines with arrows point is a different kind of attractor from the fixed point one in Figure 6. This attractor is known as a *limit cycle* or *periodic* attractor of a "pushed" swing. It is called a "limit cycle" because it represents the cyclical behavior of the oscillations of the pushed swing as a "limit" to which the systems adheres when under the sway of this attractor. It is  "periodic" because the attractor oscillates around the same values, as the swing keeps going up and down  to the same heights from the lowest point. One can tell a dynamical system is periodic if it has a repeating cycle or pattern.

As before, the lines with the arrows indicate initial conditions in the basin of attraction which are "attracted to" or drawn to the limit cycle attractor. Here is another limit cycle attractor of the dynamical system known as the van der Pol electrical oscillator:

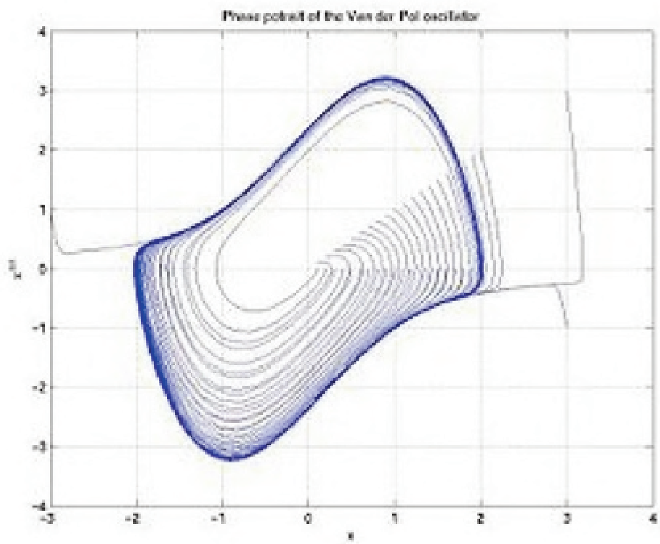


*Figure 9: Limit Cycle Attractor of the van der Pol Electrical oscillator (adapted from Dynamical System, No date).*

Because attractors are the stable pattern to which a dynamical system's dynamics are attracted, sometimes one hears the term being used in a causal sense, that is, that the attractor is "causing" the system's behavior. Some have even considered the idea of an attractor as a type of final cause, or end or "telos" in terms of Aristotle's four causes. However, it doesn't appear to make much sense to say that the attractors of Figures 6, 7, and 9 are causing the systems' behavior. For example, does it make a lot of sense to say the fixed point attractor of Figure 6 is causing the "unpushed" swing to come to a state or rest? It seems more appropriate, instead, to consider the attractor as a description of the dynamics of the dynamical system. It is important to keep in mind that phase portraits *represent* and do not cause the dynamics. Indeed, phase portraits and attractors do reveal important information about the causal elements operative in a system, for instance, that the variables are nonlinearly related to one another and so forth. That is why a quantitative and qualitative study of the geometrical, topological, and other properties of the attractors can yield deep insights into the system's dynamics.

To summarize what we have learned so far about attractors: they are spatially displayed phase portraits of a dynamical system as it changes over the course of time; thus they represent the *long-term* dynamics of the system so that whatever the initial conditions, represented as data points and their trajectories in phase space that fall within its basins of attraction, they are "attracted" to the attractor.

Yet, in his very insightful survey of the idea of an attractor, Robinson (2005) points out that in spite of its wide usage in mathematics and science, there is still no precise definition of an attractor, although many have been offered. Robinson suggests that we think about an attractor as a phase portrait that "attracts" a large set of initial conditions (starting points of the system's dynamics or behavior) and that has some sort of minimality property. That means that it is the smallest such portrait in the phase space of the system that has the property of attracting the initial conditions after any initial transient behavior has died down. The minimality requirement has the consequence that the attractor is invariant or stable. It is the stable spatial pattern in phase space that possesses the property of capturing the evolution of the system over time. As a minimal object, the attractor is also said to be *indecomposable,* that is, it cannot be split up into smaller subsets and retain its role as what dominates a dynamical system during a particular phase of its evolution.

## Chaos, Chaotic Attractors, and Strange Attractors

What is gained by switching from time series charts to representing data using phase space diagrams and their portraits and attractors may be not be great when we're only talking about the simple kind of dynamics seen above in the cases of the "unpushed" and "pushed" swings. But the real benefits of using attractor reconstructions of data becomes apparent when the system's dynamics are more complex, often making it quite difficult to recognize repeating or near repeating patterns in time series charts. This is where the discovery of technically *chaotic* systems became so important. Representing the data generated from such systems led to the remarkable discovery that seemingly random systems may in fact be deterministic yet posses random-appearing chaotic attractors. These chaotic systems only appear random but are really constituted by a very complex type of stable order, which, although not regular or repeating, nevertheless keeps the dynamical system within certain ranges of possible behavior. Whereas the phase portrait of a truly random system would eventually totally fill up the whole phase space diagram with a big black blob since no region in phase space would be preferred over another, in chaotic systems the phase portraits are very intricate structures delimiting the dynamics to only circumscribed regions in their phase diagrams.

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The late mathematical meteorologist Edward Lorenz (1993) uncovered this kind of behavior in research conducted during the early nineteen sixties using mathematical models of the weather. Lorenz noticed that a seemingly insignificant difference in a mathematical model he was using for forecasting the weather resulted in significantly different forecasts. In effect, what Lorenz discovered was the phenomena of sensitive dependence on initial conditions—the butterfly effect—in which a slight change in an initial condition might lead to vast changes in outcomes. This was a phenomenon that had been glimpsed more than a century ago by the great French mathematician Henri Poincaré in his qualitative theory of differential equations, "qualitative" in the sense that he found patterns among sets of the equations he was studying. Another eminent French mathematician, Jacques Hadamard, demonstrated something like sensitive dependence on initial conditions in a quite different mathematical context (Rosser, 2009).

What Lorenz had discovered would later be known as chaos and its graphic display in phase space as a chaotic attractor since it had the properties of being **a**-periodic, that is the trajectory of the phase portrait never repeats (never crosses itself) and is sensitively dependent on initial conditions (the "butterfly effect'). Robinson (2005) points out that an attractor built out of Lorenz's data was the first explicit example of an attractor that was neither a fixed point nor some kind of periodic orbit. Here is a diagram of a Lorenz attractor:

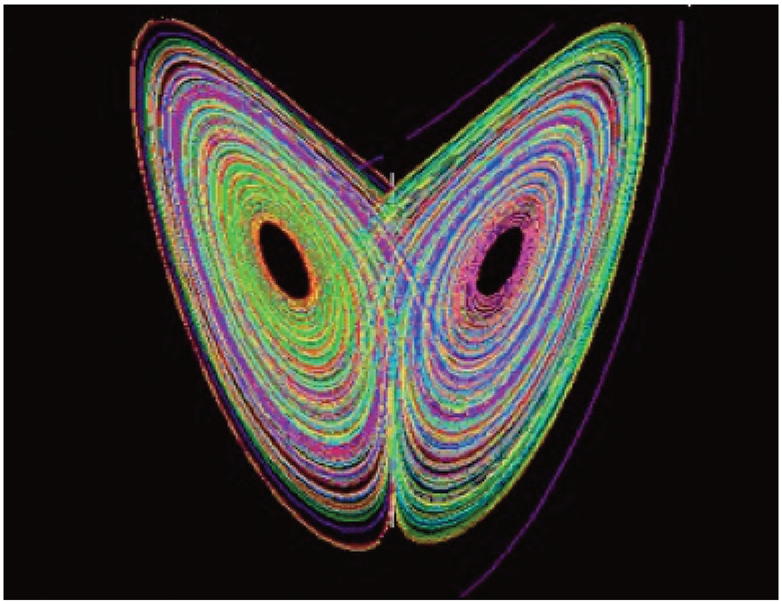

*Figure 10: Lorenz Attractor (Ecometry, No date). The phase portrait plots such variables against each other as the convective flow in the atmosphere and a vertical and a horizontal distribution of temperature. Please note that this should be visualized as three dimension since Lorenz had been employing three variables in his models.*

In Figure 10, although it appears that the trajectory in phase space crosses itself innumerable times, in actuality, the trajectory only passes arbitrarily close (remember it is three dimensional so the trajectory only appears to cross itself but really fall in-between). Today, in this context, we say that the Lorenz attractor has a fractal, that is a non-integer dimensionality. This means that even though the phase portrait appears to be merely a two-dimensional trajectory in phase space, this is simply an effect caused by the "coarse" grained nature of the graphic display and our visual abilities. If we could peer deeper and deeper into finer and finer scales of resolution, we would see that the trajectory never actually crosses but instead gets arbitrarily close and then veers of. The characteristic of getting arbitrarily close and then veering off is the spatial representation of the property of sensitive dependence on initial conditions.

After these early glimpses of chaotic behavior in dynamical systems, it took another decade for such phenomena to be named "chaos" in a strict mathematical sense, distinguished from the usual randomness connotations of the word "chaos" (see Li & Yorke, 1975; May, 1976). These were startling developments in mathematics since they demonstrated that underlying what appeared to be random time series were actually deterministic systems produced by lawful and not random operations. Moreover, there is some measure of predictability in chaotic systems because of the way the attractors of the system are constrained to particular regions of phase space. For example, if the weather is modeled as a chaotic system, so that particular states of the weather are unpredictable (for example, what temperature will it be in New York City on September 11, 2015?), it nevertheless predictable that the temperature will fall within a range, say, between 72 and 95 degrees Fahrenheit. This predictability results from how the *climate* per se can be interpreted as one of the factors that *attracts* the weather and thereby serves as a constraint on the unpredictability of the states of the weather.

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## Strange Attractors

The term and idea of a "strange attractor" came about in research by the mathematical physicists David Ruelle and Floris Takens (1971) into the phenomena of turbulence in viscous liquids, an area long considered difficult for several reasons, not the least being the mostly intractable equations used to model fluid dynamics. Ruelle and Takens came up with a mathematical means of modeling turbulence, that is certain functions that when iterated led to an attractor with strange properties, hence the name "strange attractor." One of these strange properties was that the attractor had what would later be called a fractal structure since it was composed of the product of a Cantor set with a two dimensional surface. This fractal Cantor ternary set, as seen in Figure 11, is created by repeatedly deleting the open middle thirds of a set of line segments, each new scale showing the same tripartite structure:

*Figure 11: Fractal structure of a Cantor set (Cantor Set, No date).*

Takens and Ruelle's notion of a "strange attractor" was later applied to the attractors of chaotic systems since it was generally thought that all chaotic attractors had a fractal structure as well as possessing the property of sensitive dependence on initial conditions (the butterfly effect from above). Thus "strange" and "chaotic" in relation to attractors became synonymous.

However, to complicate matters a bit, still later it was discovered that there are not only strange non-chaotic attractors that have a fractal structure but do not possess the property of sensitive dependence on initial conditions. There are also non-strange chaotic attractors that don't have a non-integer dimensionality, that is, they do not possess a fractal structure (see, Grebogi, Ott, Pelikan & Yorke, 1984; and, Awrejcewicz, & Reinhardt, 1990).

To be sure, the idea that there could be fractal, non-integer dimensions was quite a revolutionary idea and there's no doubt that its progenitor, the late great mathematician Benoit Mandelbrot (1982), has a significant place in mathematical immortality. Fractal dimensioned strange attractors can be quite beautiful as Figure 12 exhibits:

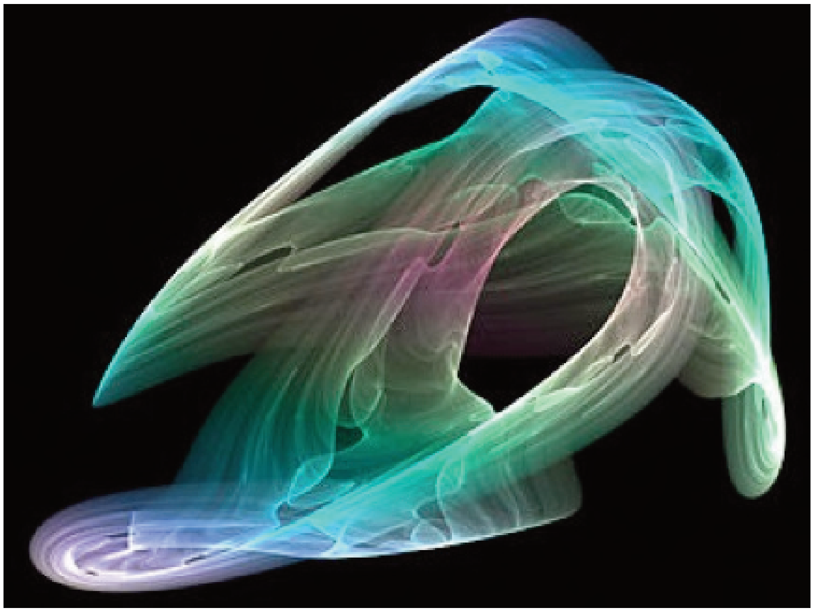

*Figure 12: Poisson Saturne Attractor (adapted from Attractor, No date) (color added to show more clearly the intricacy and beauty of the form).*

The physicist Clint Sprott has a website (Sprott, No date) where new, beautiful fractals and strange attractors are created daily. Sprott (2005) has also researched differences in preferences between scientists and artist as to their appreciation of strange attractors. Interestingly, scientists tend to prefer less complex strange attractors while artists tend to prefer the more complex varieties.

## Bifurcations

Dynamical systems undergo change in two different modes. The first we can call ***intra***-attractor change, since the changes are limited to what is circumscribed by the reigning attractor(s). Intra-attractor change does not disrupt stability, demonstrated by the fact that the system has a stable attractor(s). However a dynamical system can also undergo a more radical type of change when the attractor(s) themselves change, an ***inter***-attractor change known as *bifurcation.* Bifurcation(s) result when certain parameters on the dynamical equations, that is conditions affecting the system, reach critical thresholds (see May, 1976). When bifurcations takes place, the phase portrait undergoes a qualitative shift to a new kind of spatial patterning, that is, what was previously describing the stability itself undergoes an instability resulting in new stable configurations in state space.

There are different routes to the occurrence of bifurcation, one of them being the Ruelle-Takens scenario described above (see also Bifurcation Theory, No date) in which a periodic attractor bifurcates into a torus (a donut like shape) and the torus into a strange attractor. Another route to bifurcation is the period doubling one studied by the physicist Mitchell Feigenbaum who discovered the constant that bears his name (see Williams, 1997). In the latter scenario, the period of the attractors keeps doubling until the attractors pass into a chaotic attractor.

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## Chaotic Attractors in Heart Beat Interval Research

Below is an example of the usefulness of using the mathematical devices of phase space and attractors in physiological research. Building on the ground-breaking work of Ari Goldberger (see Goldberger, 1996) in studying the nonlinear dynamics of heart beat intervals, Osaka, et al. (2003) investigated the role of sympathetic nerve activity on heart beat intervals by studying rats with normal or high blood pressure (I guess the rat race really got to them!). These researchers, assuming "heartbeat intervals are determined by sympathetic nerve activity and blood pressure in a complex interaction that involves the brainstem and feedback loops," went on to examine the details of the interaction for low-frequency oscillations. Figure 13 depicts time series charts of their data:

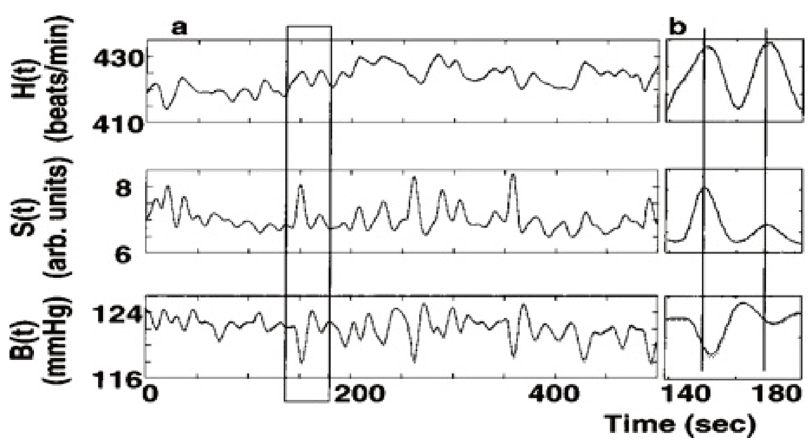


*Figure 13: Time Series of Heart Rate (H(t)), Renal Sympathetic Nerve Activity (S(t)), and Blood Pressure (B(t)) in Rats with Normal or High Blood Pressure (adapted from Osaka, et. al., 2003).*

One thing noticeable by comparing these time series charts is the timing of extrema (seen in the narrow vertical window), that is, troughs and valleys at same time in renal sympathetic activity and blood pressure.

Not content with relying on time series charts, the authors transposed their data into phase diagrams whose phase portrait indicate the existence of low dimensional chaotic attractors seen in Figures 14 and 15:

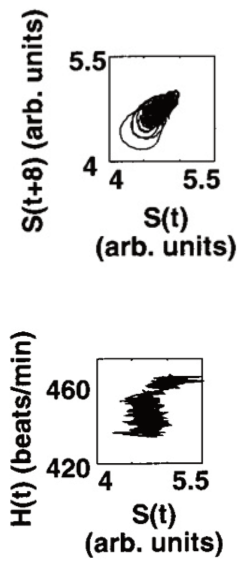


*Figure 14: At the top: Phase Portrait of Renal Sympathetic Nerve Activity against Delayed or Lagged Time Version of Itself; At the bottom: Heart Rate Intervals Plotted Against Renal Sympathetic Activity (adapted from Osaka, et. al., 2003).*

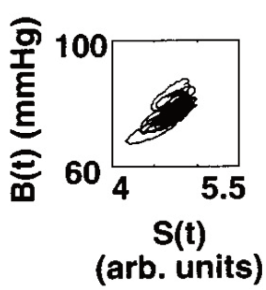


*Figure 15: Phase Portrait of Renal Sympathetic Activity in Relation to Blood Pressure (adapted from Osaka, et al., 2003).*

Each of the phase portraits in Figures 14 and 15 indicate the presence of low dimensional chaotic attractors. A "low dimensional" chaotic attractor is one where there is less variation, or, in other words, more correlation among the variables. This suggests that the low-frequency blood pressure oscillations actually do arise from sympathetic nerve activity and thus the researchers concluded that sympathetic nerve activity leads to heartbeat interval and blood pressure changes. "This bolsters the view that sympathetic nerve activity may play a causative role in hypertension" (p. 041915-3). This conclusion illustrates how attractor (re)construction as a research tool can be used to probe possible causal mechanisms at work in the system, but that the attractor itself is not the causative agent. It is from findings like these that certain diseases are now understood as "dynamical diseases" meaning that their temporal phasing can be a key to understanding pathological conditions.

## Conclusion

One of the powerful appeals of the use of phase space, phase portraits, and attractor reconstructions is their graphic vividness. This goes along with how changes over time in a time series chart are transformed into spatial patterns in phase space, which are then more accessible in many ways for gaining insight into the nonlinear dynamics of such systems. It is no accident that the study of such complex systems has arisen at the same time as the micro-processor, an invention that has heralded much of the mathematical underpinnings of contemporary complexity theory.

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